Fractional Dynamics of Individuals In Complex Networks

presented at

Fractional PDEs:
Theory, Algorithms & Applications

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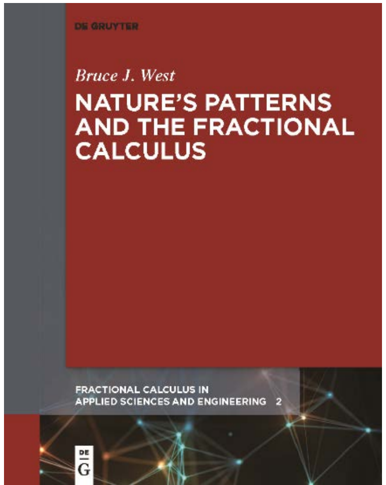

2017

Why fractional calculus?

- Why a fractional calculus?
  - new ways of thinking
  - complexity
  - dynamics and fractals
- Complex dynamic networks
  - critical behavior
  - consensus
- Fractional dynamics
  - simple fractional operators
  - fractional rate equation
  - Mittag-Leffler function
  - phase space fractional equations
- Conclusions

Classical Geometry

Fractal Geometry

Fractal Dynamics

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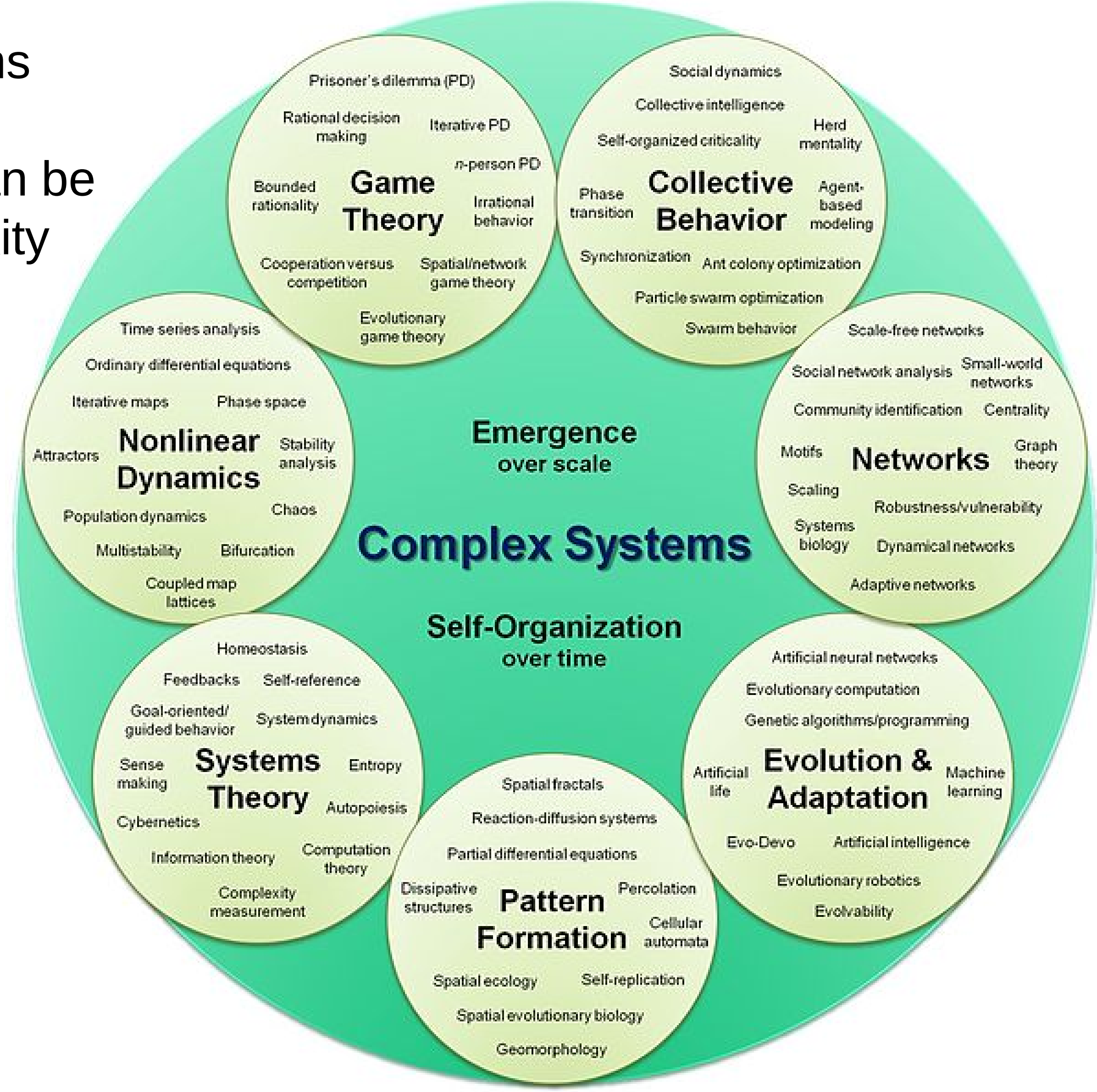
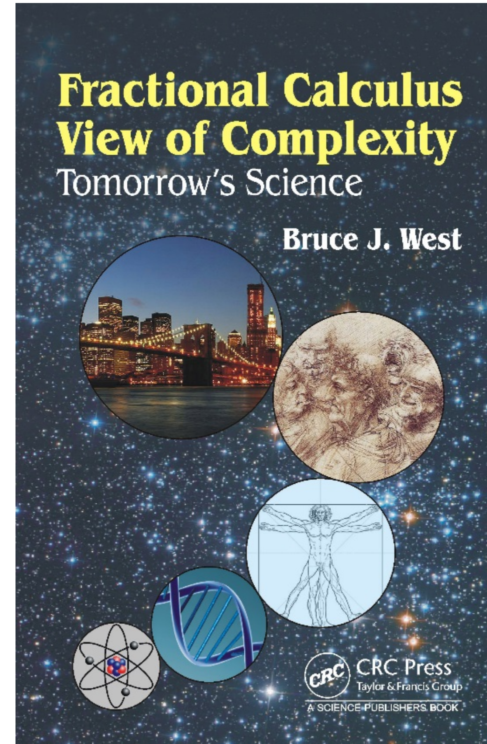
Complex systems

- Army research problems
- All research barriers can be traced back to complexity
  - Chaos
  - Statistics
  - Multiple scales
  - Criticality
- How do we understand complexity?



Fractional thinking

2016



- Introduce in-between thinking:
  - between integers are infinity of non-integers
  - between integer-order moments are fractional moments
  - between integer dimensions are fractal dimensions
  - between integer Taylor series are fractional Taylor series
  - between integer-valued operators are fractional-order operators

- This tutorial is on how the fractional calculus provides insight into complex phenomena.

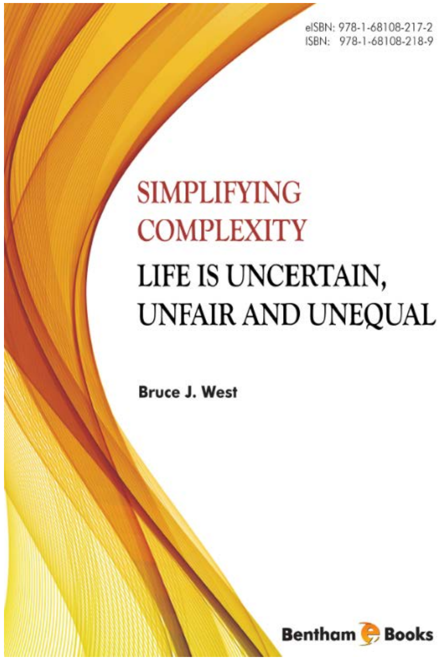
- Complexity is emphasized, which highlights the inability of traditional analytic functions to satisfactorily characterized the rich structure of complex dynamic phenomena in both space and time.

Out with the old…

….NEW WAY OF THINKING….

old

new

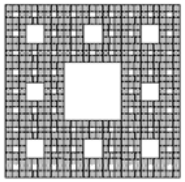


2016

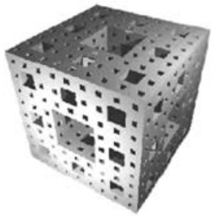
Why is the fractional calculus entailed by complexity?

Fractal dimensions

…from integer to non-integer…

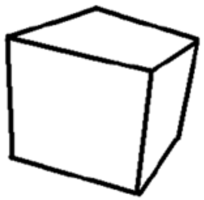


$D = 1.89$



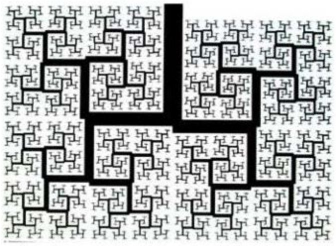
$D = 2.73$

$D = 3$

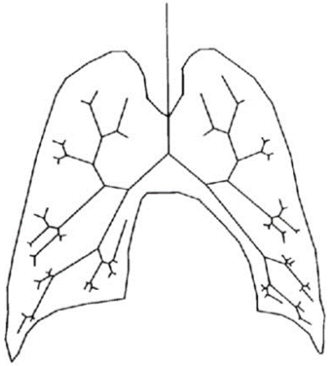


$D = 2$

$D = 1.26$



$D = 1$



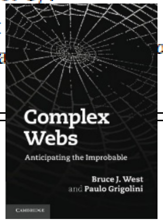
How pervasive are non-integer phenomena?

Fractal processes ?

Empirical Power Laws

Discipline	Law's name	Form of law
Anthropology		
1913 [4]	Auerbach	Pr(city size rank r) $\propto 1/r$
1998 [65]	War	Pr(intensity $> I$) $\propto 1/I^{\alpha}$
1978 [86]	$1/f$ Music	Spectrum(f) $\propto 1/f$
Biology		
1992 [87]	DNA sequence	Symbol spectrum(frequency f) $\propto 1/f^{\alpha}$
2000 [49]	Ecological web	Pr(k species connections) $\propto 1/k^{1.1}$
2001 [35]	Protein	Pr(k connections) $\propto 1/k^{2.4}$
2000 [34]	Metabolism	Pr(k connections) $\propto 1/k^{2.2}$
2001 [40]	Sexual relations	Pr(k relations) $\propto 1/k^{\alpha}$
Botany		
1883 [64]	da Vinci	Branching; $d_0^{\alpha} = d_1^{\alpha} + d_2^{\alpha}$
1922 [101]	Willis	No. of genera(No. of species N) $\propto 1/N^{\alpha}$
1927 [51]	Murray	$d_0^{2.5} = d_1^{2.5} + d_2^{2.5}$
Economics		
1897 [56]	Pareto	Pr(income x) $\propto 1/x^{1.5}$
1998 [24]	Price variations	Pr(stock price variations x) $\propto 1/x^{3}$
Geophysics		
1894 [55]	Omori	Pr(aftershocks in time t) $\propto 1/t$
1933 [67]	Rosen–Rammler	Pr(No. of ore fragments $<$ size r) $\propto r^{\alpha}$
1938 [44]	Korčak	Pr(island area $A > a$) $\propto 1/a^{\alpha}$
1945 [31]	Horton	No. of segments at n/No. of segments at $n+1$ = constant
1954 [26]	Gutenberg–Richter	Pr(earthquake magnitude $< x$) $\propto 1/x^{\alpha}$
1957 [27]	Hack	River length $\propto$ (basin area)$^{\alpha}$
1977 [44]	Richardson	Length of coastline $\propto$ 1/(ruler size)$^{\alpha}$
2004 [84]	Forest fires	Frequency density(burned area A) $\propto 1/A^{1.38}$
Information theory		
1999 [32]	World Wide Web	Pr(k connections) $\propto 1/k^{1.94}$
1999 [19]	Internet	Pr(k connections) $\propto 1/k^{\alpha}$

Discipline	Law's name	Form of law
Physics		
1918 [70]	$1/f$ noise	Spectrum(f) $\propto 1/f$
2002 [25]	Solar flares	Pr(time between flares t) $\propto 1/t^{2.14}$
2003 [69]	Temperature anomalies	Pr(time between events t) $\propto 1/t^{2.14}$
Physiology		
1959 [61]	Rall	Neurons; $d_0^{1.5} = d_1^{1.5} + d_2^{1.5}$
1963 [76]	Mammalian vascular network	Veins and arteries; $d_0^{2.7} = d_1^{2.7} + d_2^{2.7}$
1963 [90]	Bronchial tree	$d_0^{3} = d_1^{3} + d_2^{3}$
1973 [48]	McMahon	Metabolic rate(body mass M) $\propto M^{0.75}$
1976 [103]	Radioactive clearance	Pr(isotope expelled in time t) $\propto 1/t^{\alpha}$
1987 [93]	West–Goldberger	Airway diameter(generation n) $\propto 1/n^{1.25}$
1991 [30]	Mammalian brain	Surface area $\propto$ volume$^{0.90}$
1992 [77]	Interbreath variability	No. of breaths(interbreath time t) $\propto 1/t^{2.16}$
1993 [58]	Heartbeat variability	Power spectrum(frequency f) $\propto f$
2007 [23]	EEG	Pr(time between EEG events) $\propto 1/t^{1.61}$
2007 [13]	Motivation and addiction	Pr(k behavior connections) $\propto 1/k^{\alpha}$
Psychology		
1957 [75]	Psychophysics	Perceived response(stimulus intensity x) $\propto x^{\alpha}$
1963 [71]	Trial and error	Reaction time(trial N) $\propto 1/N^{0.91}$
1961 [29]	Decision making	utility(delay time t) $\propto 1/t^{\alpha}$
1991 [3]	Forgetting	Percentage correct recall(time t) $\propto 1/t^{\alpha}$
2001 [20]	Cognition	Response spectrum(frequency f) $\propto 1/f^{\alpha}$
2009 [37]	Neurophysiology	Pr(phase-locked interval $< \tau$) $\propto 1/\tau^{\alpha}$
Sociology		
1926 [41]	Lotka	Pr(No. of papers published rank r) $\propto 1/r^{2}$
1949 [104]	Zipf	Pr(word has rank r) $\propto 1/r$
1963 [16]	Price	Pr(citation rank r) $\propto$
1994 [8]	Urban growth	Population density(ra α
1998 [88]	Actors	Pr(k connections) $\propto$



Complex Webs: Anticipating the Improbable, B.J. West and P. Grigolini, Cambridge (2011).

Fractional PDEs

- There are a number of ways to derive fractional PDEs for PDFs that are fractional in both space and time:
  - Using the continuous time random walk (CTRW): Metzler & Klafter, *Phys. Rept*. **339**,1 (2000); Shlesinger, West & Klafter, *Phys. Rev. Lett*. **28**, 1100 (1987).
  - Averaging over chaotic trajectories: Zaslavsky, *Phys. Rept*. **371**, 461 (2002)
  - Subordination arguments: Pramukkul,, Svenkeson, Grigolini, Bologna & West, *Adv. Math. Phys*. 2013

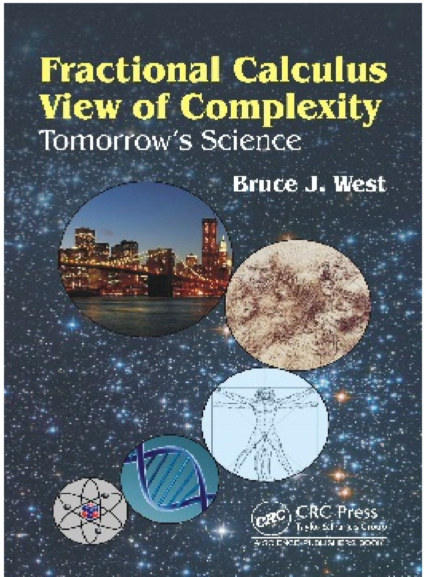
$$\partial_t^{\alpha}[P(x,t)] = K_{\beta} D_{|x|}^{\beta}[P(x,t)]$$

- The solution to this fractional PDE has the scaling form:

$$P(x,t) = \frac{1}{t^{\mu}} F\left(\frac{x}{t^{\mu}}\right) \;\; ; \;\; \mu = \frac{\alpha}{\beta}$$

Uchaikin, *Int. J. Theor. Phys*. **39**, 3805, 2000.

- We apply subordination theory to dynamic groups.

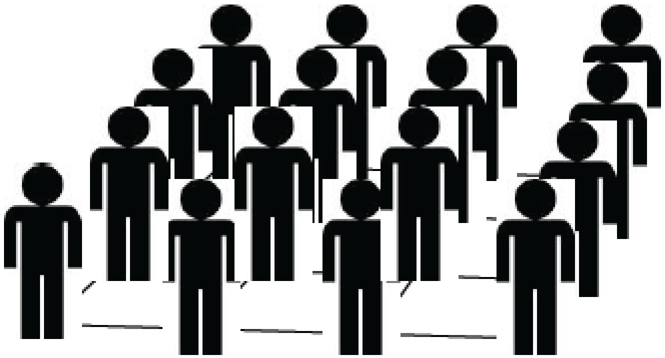


2016

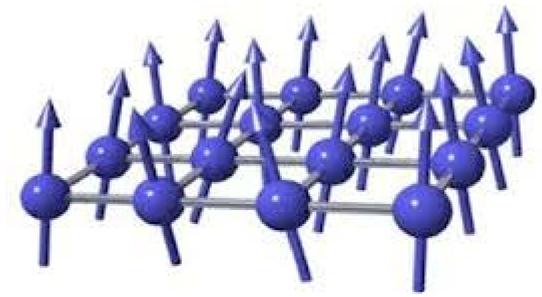
Complex Network Dynamics

- One approach to the fractional calculus

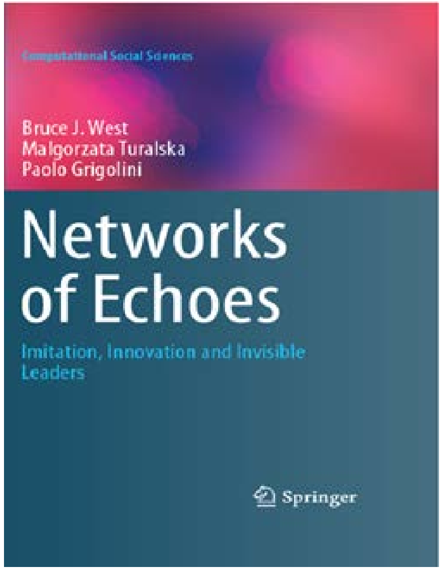
Human network



Network model



- Two-state master equation decsion making model (DMM)
- DMM is member of Ising universality class
  - phase transitions to consensus
  - scaling behavior
  - temporal complexity
- How does the network dynamics influence individual dynamics?



2014

Subordination 1

- Subordination models numerical integration of individual opinion *s(n)* in discrete operational time *n:*

$$s(n+1)=(1-\Delta\tau g)s(n) \;\Rightarrow\; s(n)=(1-\Delta\tau g)^n s_0$$

- This is the time experienced by the individual and for $g\Delta\tau \ll 1$ is a Poisson proces

- The influence of network dynamics on individual in chronological time *t* is (*The Organization Man*)

$$\langle s(t)\rangle=\sum_{n=0}^{\infty}\int_0^t \psi_n(t')\Psi(t-t')s(n)dt'$$

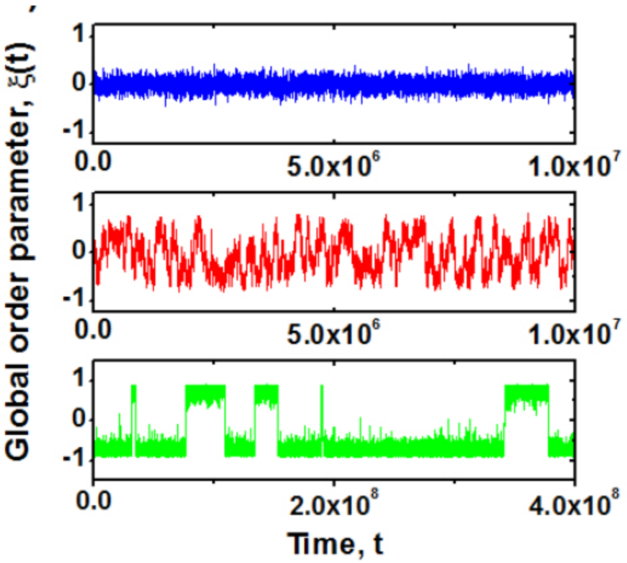
Probability density of last of *n* events occurs in time *(0,t')*

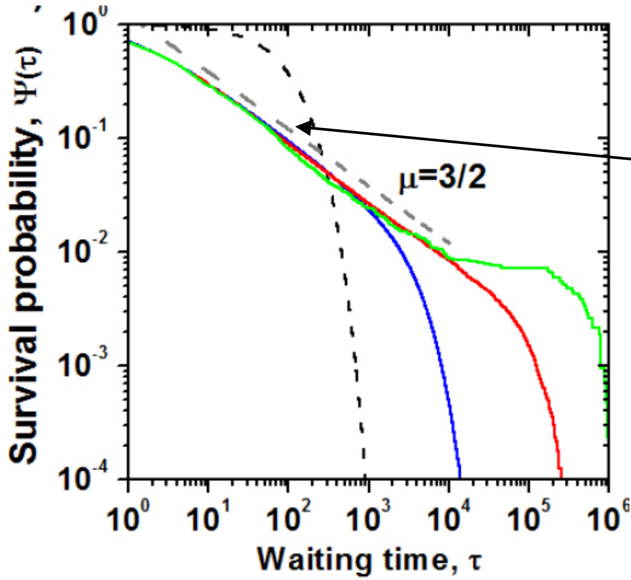
Probability no event occurs in *(t-t')*

Pramukkul, Svenkeson, Grigolini, Bologna & West, *Advances in Mathematical Physics 2013, Article ID 498789* (2013).

Subordination 2

• The waiting-time distribution and survival probability are taken from numerics.





$$\Psi(t) = \left(\frac{T}{T+t}\right)^{\mu-1} e^{-\epsilon t}$$

• Solve the subordination equation using Laplace transforms to obtain fractional differential equation for average individual opinion:

$$(\partial + \boldsymbol{\epsilon})^{\alpha}\langle s(t)\rangle = -\gamma^{\alpha}\langle s(t)\rangle \;\; ; \;\; \alpha = \mu - 1$$

• This is the predicted average dynamics of the single individual within the social network.

Subordination 3

- Fractional dynamics of individuals in complex networks; Turalska& West, *Frontiers in Physics* (under review)

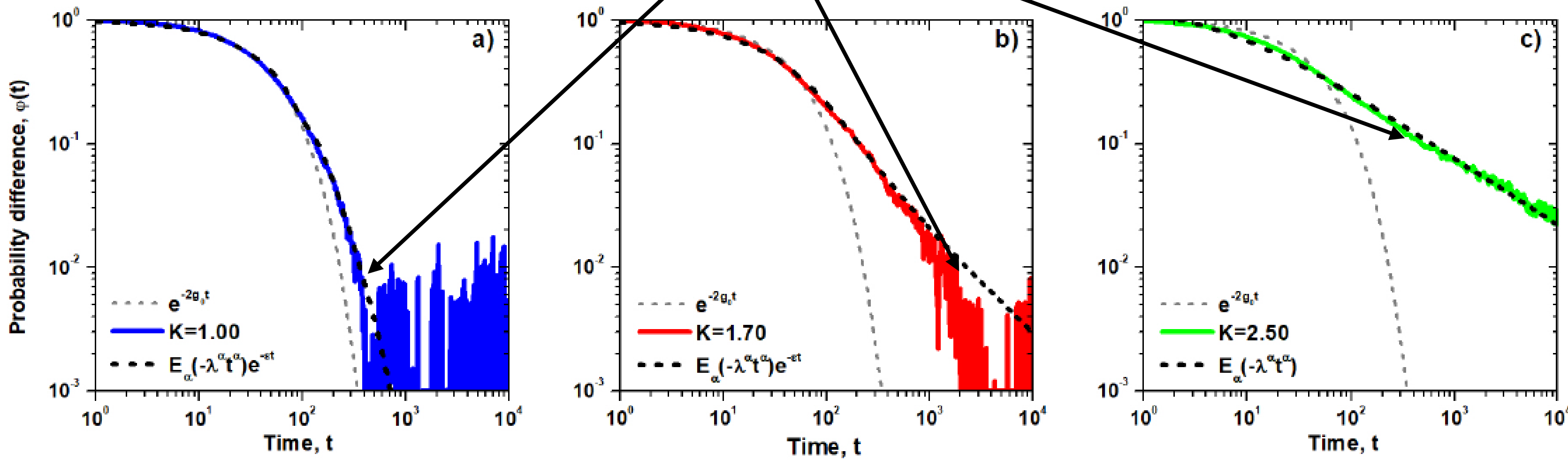
$$(\partial + \epsilon)^{\alpha}\langle s(t)\rangle = -\lambda^{\alpha}\langle s(t)\rangle \;;\; \alpha = \mu - 1$$

- Solution to FDE is the tempered Mittag-Leffler function

$$\langle s(t)\rangle = E_{\alpha}(-(\lambda t)^{\alpha})e^{-\epsilon t}$$

Network Effect

$r^2 \geq 0.99$



$\alpha = 0.91$ $K \leq K_C$ $\alpha = 0.81$ $K = K_C$ $\alpha = 0.53$ $K \geq K_C$

Tale’s End

Conclusions

• FC is a strategy for systematically addressing complexity-induced research challenges.

• FC entails a new way of thinking, as different as geometry is from number theory.

• FC provides a framework for the dynamics of scale-free complex networks.

• The influence of a network on an individual is described by a stochastic FDE.

• Network dynamics transforms a capricious decision maker into a thoughtful one.

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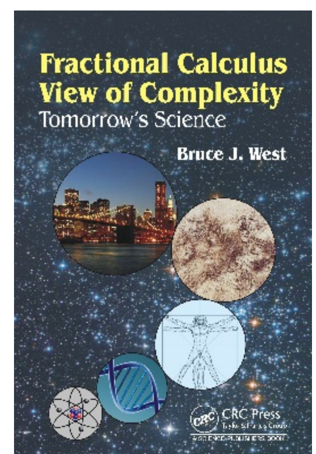

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